

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.  
Birzeit University, Palestine, 2015

# Counting

9.1 Basics of Probability and Counting

**9.2 Possibility Trees and the Multiplication Rule**

9.3 Counting Elements of Disjoint Sets: Addition Rule

9.5 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed



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**Acknowledgement:**

This lecture is based on (but not limited to) to chapter 9 in “Discrete Mathematics with Applications by Susanna S. Epp (3<sup>rd</sup> Edition)”.

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# Counting

## 9.2 Possibility Trees and the Multiplication Rule

In this lecture:

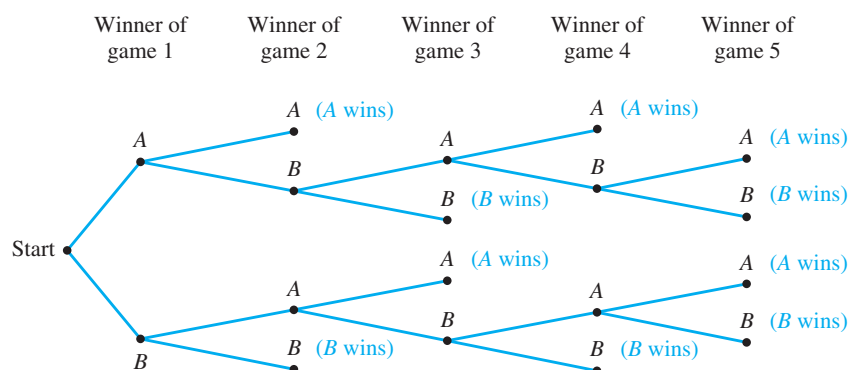
- ☒ Part 1: **Possibility Trees**
- ☐ Part 2: **Multiplication Rule**
- ☐ Part 3: **Permutations**

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## Possibility Trees

Teams  $A$  and  $B$  are to play each other repeatedly until one wins two games in a row or a total of three games

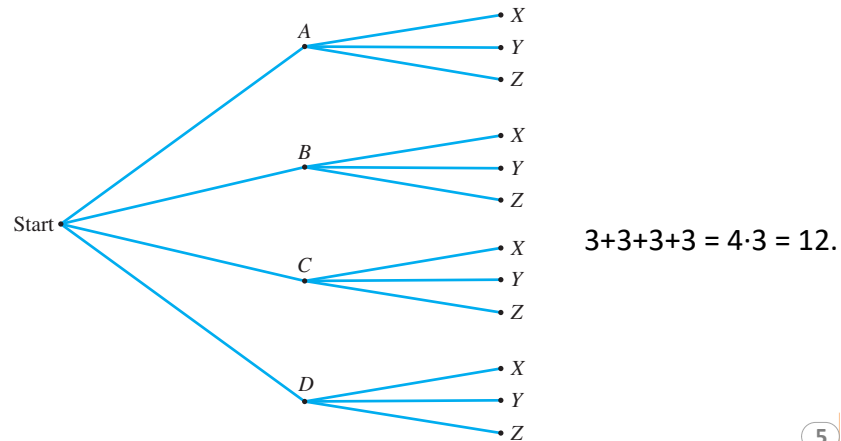
*How many ways can the tournament be played?*



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### Possibility Trees

We have 4 computers (A,B,C,D) and 3 printers (X,Y,Z). Each of these printers is connected with each of the computers. *Suppose you want to print something through one of the computers, How many possibilities for you have?*



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### Possibility Trees

A person buying a personal computer system is offered a choice of 3 models of the basic unit, 2 models of keyboard, and 2 models of printer.

*How many distinct systems can be purchased?*

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## Possibility Trees

Notices that representing the possibilities in a tree structure is a useful tool for tracking all possibilities in situations in which events happen in order.

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## 9.2 Possibility Trees and the Multiplication Rule

In this lecture:

☐ Part 1: **Possibility Trees**



☒ Part 2: **Multiplication Rule**

☐ Part 3: **Permutations**

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## The Multiplication Rule

### Theorem 9.2.1 The Multiplication Rule

If an operation consists of  $k$  steps and

the first step can be performed in  $n_1$  ways,

the second step can be performed in  $n_2$  ways [regardless of how the first step was performed],

$\vdots$

the  $k$ th step can be performed in  $n_k$  ways [regardless of how the preceding steps were performed],

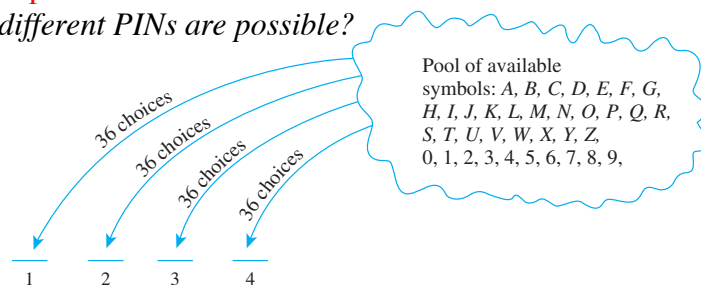
then the entire operation can be performed in  $n_1 n_2 \cdots n_k$  ways.

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## Counting Example 1

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the 10 digits, **with repetition allowed**.

How many different PINs are possible?



Step 1: Choose the first symbol.

Step 2: Choose the second symbol.

Step 3: Choose the third symbol.

Step 4: Choose the fourth symbol.

$$36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1,679,616 \text{ PINs in all.}$$

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### Counting Example 1

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the 10 digits, **with repetition not allowed**.

*How many different PINs are possible?*

$$36 \cdot 35 \cdot 34 \cdot 33 = 1,413,720$$

*what is the probability that a PIN chosen at random contains no repeated symbol?*

$$\frac{1,413,720}{1,679,616} \approx .8417$$

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### Counting Example 2

**for**  $i := 1$  **to** 4

**for**  $j := 1$  **to** 3

        [Statements in body of inner loop.

        None contain branching statements  
        that lead out of the inner loop.]

**next**  $j$

**next**  $i$

How many  
times this  
statement will  
be executed?

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### Counting Example 3

Suppose  $A_1, A_2, A_3$ , and  $A_4$  are sets with  $n_1, n_2, n_3$ , and  $n_4$  elements, respectively.

How many elements in  $A_1 \times A_2 \times A_3 \times A_4$

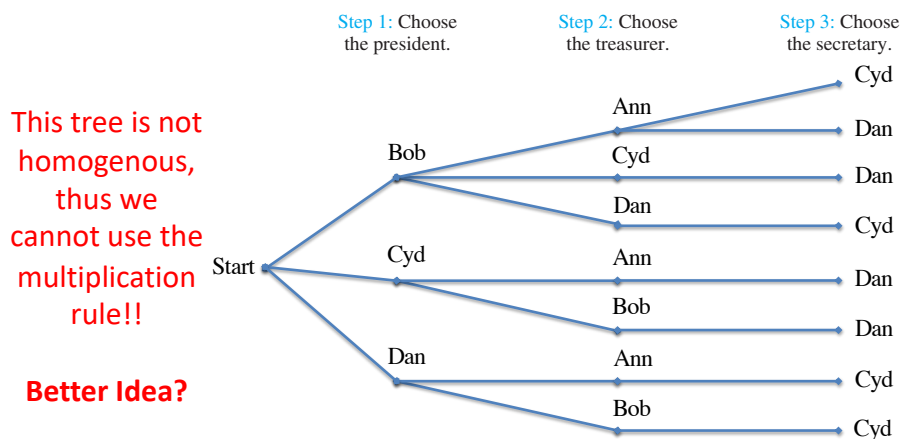
Solution: Each element in  $A_1 \times A_2 \times A_3 \times A_4$  is an ordered 4-tuple of the form  $(a_1, a_2, a_3, a_4)$

By the multiplication rule, there are  $n_1 n_2 n_3 n_4$  ways to perform the entire operation. Therefore, there are  $n_1 n_2 n_3 n_4$  distinct 4-tuples in  $A_1 \times A_2 \times A_3 \times A_4$

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### Counting Example 4

تم انتخاب أربعة طلاب لنادي الكلية (Ann, Bob, Cyd, Dan): نريد اختيار رئيس، أمين صندوق، وسكرتير. لا يمكن لـ Ann ان تكون رئيساً، والسكرتير اما ان يكون Dan او Cyd كم تشكيلة ممكنة؟



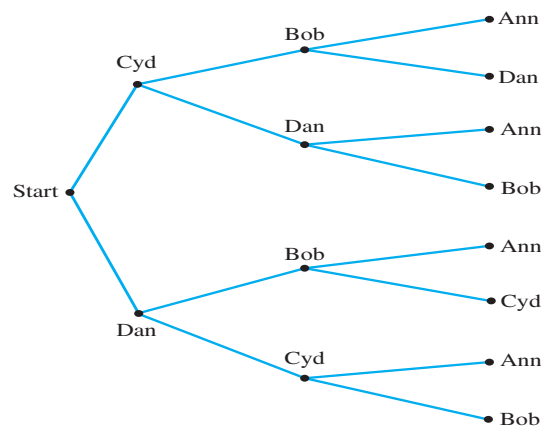
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### Counting Example 4

تم انتخاب أربعة طلاب لنادي الكلية (Ann, Bob, Cyd, Dan): نريد اختيار رئيس، أمين صندوق، وسكرتير. لا يمكن لـ Ann ان تكون رئيساً، والسكرتير اما ان يكون Dan او Cyd كم تشكيلة ممكنة؟

Step 1: Choose the secretary.    Step 2: Choose the president.    Step 3: Choose the treasurer.

We should be smart to represent our problem in a way to be able to use the multiplication rule



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## Counting

### 9.2 Possibility Trees and the Multiplication Rule

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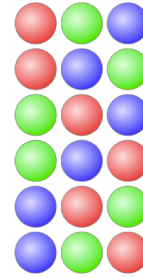
- ☐ Part 1: **Possibility Trees**
- ☐ Part 2: **Multiplication Rule**
- ☒ Part 3: **Permutations**

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## Permutations

التباديل: عدد التشكيلات الممكنة لمجموعة جزئية من  
العناصر منتقاة من مجموعة كلية من العناصر مع مراعاة  
لأهمية تسلسل العناصر في تشكيلات المجموعة الجزئية  
كم كلمة من خمس حروف ممكن ان نكون اذا كان لدينا  
عشرة حروف؟



كانت القاعدة التي تمكن من حساب عدد التباديل لمجموعة ما، معروفة لدى  
الهنديين على الأقل في حوالي عام 1150م.

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## Permutations

A **permutation** of a set of objects is an ordering of the objects in a row.

For example, the set of elements  $\{a, b, c\}$  has six permutations.

$abc \quad acb \quad bac \quad bca \quad cab \quad cba$

Generally, given a set of  $n$  objects, how many permutations does the set have? Imagine forming a permutation as an  $n$ -step operation:

**Step 1:** Choose an element to write **first**.

**Step 2:** Choose an element to write **second**

...

**Step  $n$ :** Choose an element to write  **$n$ th**.

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## Permutations

by the multiplication rule, there are

$$n(n-1)(n-2) \cdots 2 \cdot 1 = n!$$

ways to perform the entire operation.

### Theorem 9.2.2

For any integer  $n$  with  $n \geq 1$ , the number of permutations of a set with  $n$  elements is  $n!$ .

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## Example 1

How many ways can the letters in the word *COMPUTER* be arranged in a row?

$$8! = 40,320.$$

How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

$$7! = 5,040.$$

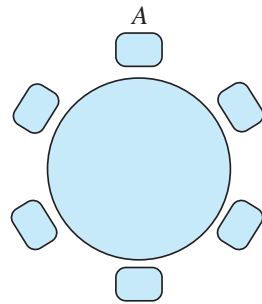
If letters of the word *COMPUTER* are randomly arranged in a row, what is the probability that the letters *CO* remain next to each other (in order) as a unit?

$$\frac{5,040}{40,320} = \frac{1}{8} = 12.5\%.$$

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## Example 2

كيف يمكن توزيع ستة دبلوماسيين حول طاولة مستديرة



Five other diplomats to be seated:  $B, C, D, E, F$

لأنها مستديرة، ثبت واحدة، ويبقى خمسة يمكن تبديلها

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## Permutations of Selected Elements

Given the set  $\{a, b, c\}$ , there are six ways to select two letters from the set and write them in order.

$ab \ ac \ ba \ bc \ ca \ cb$

Each such ordering of two elements of  $\{a, b, c\}$  is called a *2-permutation* of  $\{a, b, c\}$ .

أي مجموع التباديلات التي يمكن أن ننتقي بها أفراد المجموعة مع مراعاة الترتيب.

### • Definition

An  **$r$ -permutation** of a set of  $n$  elements is an ordered selection of  $r$  elements taken from the set of  $n$  elements. The number of  $r$ -permutations of a set of  $n$  elements is denoted  $P(n, r)$ .

How many permutations in  $P(n, r)$  ?

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## Permutations of Selected Elements

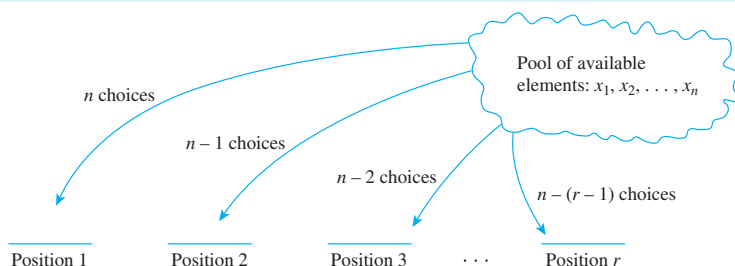
### Theorem 9.2.3

If  $n$  and  $r$  are integers and  $1 \leq r \leq n$ , then the number of  $r$ -permutations of a set of  $n$  elements is given by the formula

$$P(n, r) = n(n-1)(n-2) \cdots (n-r+1) \quad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \quad \text{second version.}$$



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## Example 3

- a. Evaluate  $P(5, 2)$ .

$$P(5, 2) = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 20$$

- b. How many 4-permutations are there of a set of 7 objects?

$$P(7, 4) = \frac{7!}{(7-4)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$$

- c. How many 5-permutations are there of a set of 5 objects?

$$P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120.$$

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### Example 4

How many different ways can 3 of the letters of the word *BYTES* be chosen and written in a row?

$$P(5, 3) = \frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 5 \cdot 4 \cdot 3 = 60.$$

How many different ways can this be done if the first letter must be *B*?

$$P(4, 2) = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 4 \cdot 3 = 12.$$



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### Example 5

Prove that for all integers  $n \geq 2$ ,

$$P(n, 2) + P(n, 1) = n^2.$$

$$P(n, 2) = \frac{n!}{(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = n(n-1)$$

and

$$P(n, 1) = \frac{n!}{(n-1)!} = \frac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = n.$$

Hence

$$P(n, 2) + P(n, 1) = n \cdot (n-1) + n = n^2 - n + n = n^2,$$

(26)